public class BSTWithLRUC<K extends Comparable<? super K>>
    implements CountMapWithLRUC<K>

public class ListOfCountMapsWithLRUC<K extends Comparable<? super K>>
    implements CountMap<K>

    private List<CountMapWithLRUC<K>> trees;
rooted tree: free tree w/ one node chosen as root

rooted tree: a connected acyclic graph with a root

free tree: a connected acyclic graph with no root
(free) tree: undirected, acyclic, connected graph

TFAE:

1) $G$ is a free tree (acyclic and connected)
2) $u,v \in G.V$, unique simple path $u \rightarrow v$
3) $G$ is connected, but removing any edge disconnects
4) $G$ is connected and $|E| = |V| - 1$
5) $G$ is acyclic and $|E| = |V| - 1$
6) $G$ is acyclic but adding any missing edge results in a cycle
(free) tree: undirected, acyclic, connected graph

**TFAE:**
1) \( G \) is a free tree (acyclic and connected)

- Suppose \( G \) is a free tree
  - \((G,v) \in G\)
    - Show at least one path between \( u \) and \( v \)
      - \( G \) is a tree hence connected since \( u \approx v \) def. connected

- Show at most one path \( u \approx v \)
  - Suppose that are two distinct paths \( u \approx v \)

  ![Diagram](image)
  - Then there is a cycle in \( G \)
  - So can be \( \geq 2 \) distinct path

2) \( \forall u,v \in V, \exists \) unique simple path \( u \approx v \)

3) \( G \) is connected, but removing any edge disconnects

  - We will show \( |E| \leq |V| - 1 \) (we already know connected \( \Rightarrow |E| \geq |V| - 1 \))
  - Induction on one of the graph
  - Base case: Suppose \( G \) has 1 vertex and removing any edge disconnects.
    - \( |E| = 0 \leq |V| - 1 \)
  - Ind step: Suppose \( G \) has \( k \) vertices and all smaller graphs satisfy \( 3 \Rightarrow 4 \)
    - Take \( G \), remove any edge, leaving \( k \) connected components \( C_1, \ldots, C_m \)
    - Each \( C_i \) is smaller than \( G \) and removing any edge from \( C_i \) disconnects \( C_i \)
  - Ind hyp. applies to each \( C_i \), so \( |E_i| \leq |V_i| - 1 \)

  \[ |E| = |E_1| + \ldots + |E_m| + 1 \leq |V_1| - 1 + \ldots + |V_m| - 1 + 1 \]
  \[ = |V_1| - 1 + \ldots + |V_m| - 1 - 1 + 1 \]
  \[ = |V| - m + 1 \leq |V| - 1 \text{ since } m \geq 2 \text{ and } m \leq |V| - 1 \]

4) \( G \) is connected and \( |E| = |V| - 1 \)

5) \( G \) is acyclic and \( |E| = |V| - 1 \)

6) \( G \) is acyclic but adding any missing edge results in a cycle